# Chiral polytopes whose smallest regular cover is a polytope

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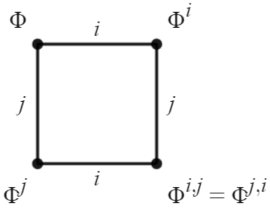
# Maniplexes

An *n-maniplex*[1] is an $n$-valent connected simple graph, whose nodes we call *flags*, such that:

1. The edges are colored $0, 1, \ldots, n-1$,
2. Each flag is incident to one edge of each color,
3. If $|i - j| > 1$, then colors $i$ and $j$ 'commute'.
   (The edges of colors $i$ and $j$ form 4-cycles.)




[1] Steve Wilson, *Maniplexes: Part 1: maps, polytopes, symmetry and operators*, 2012

# Maniplexes, maps, and polytopes

1. The flag graph of any polytope or map is a maniplex.
2. Every 3-maniplex corresponds to a map on a closed surface.
3. An $n$-maniplex is assembled from $(n-1)$-maniplexes, glued together along isomorphic $(n-2)$-maniplexes.

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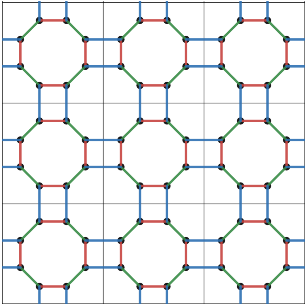
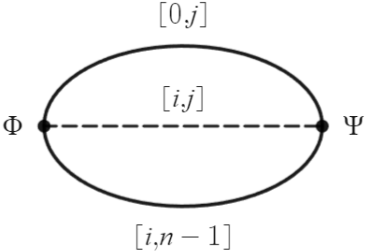


The flag graph of $\{4,4\}_{(3,0)}$

# Polytopes

An *abstract n-polytope* is an $n$-maniplex that satisfies an additional *path intersection property*[2]: that for every pair of flags $\Phi$ and $\Psi$, if there is a path between them using colors in $[0, j]$ and a path using colors in $[i, n-1]$, then there is a path that only uses colors in $[i, j]$.




[2] Jorge Garza-Vargas and Isabel Hubard, *Polytopality of maniplexes*, 2018

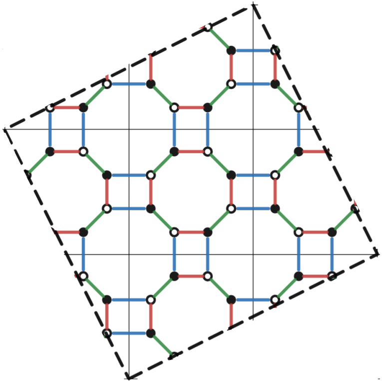
# Maniplex automorphisms

A *maniplex automorphism* is a graph automorphism that preserves the edge-labels. We denote the automorphism group by $\Gamma(\mathcal{P})$.

A maniplex is *regular* or *reflexible* if its automorphism group acts transitively on its flags.

# Chiral maniplexes

A maniplex is *chiral* if the action of the automorphism group on flags has two orbits, such that adjacent flags are always in different orbits.



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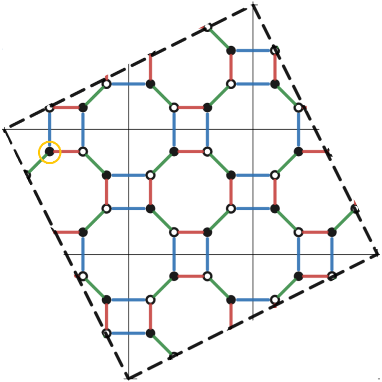

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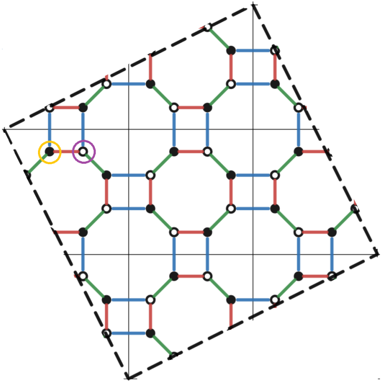

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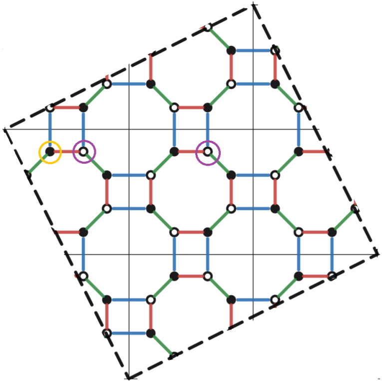

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# Facets and vertex-figures

The *facets* of an $n$-maniplex are the connected components after removing all edges of label $n$. These are $(n-1)$-maniplexes.

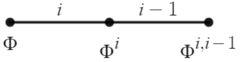
The *vertex-figures* are the connected components after removing all edges of label 0 and shifting all remaining labels down by 1. These are also $(n-1)$-maniplexes.

The facets and vertex-figures of a chiral maniplex are either chiral or regular.

# Rotation groups

If you pick a *base flag* $\Phi$ of a chiral $n$-maniplex, then there are unique automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ such that:

1. $\Phi\sigma_i = \Phi^{i,i-1}$




2. $\sigma_1, \ldots, \sigma_{n-1}$ generate the automorphism group,
3. For each $i < j$, these automorphisms satisfy $(\sigma_i \sigma_{i+1} \cdots \sigma_j)^2 = 1$.

We'll call any group $\langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ that satisfies the third item above a *string rotation group (of rank n)*.

# String $C^+$-groups

A string rotation group is a *string $C^+$-group* if $n = 2$ (and $\sigma_1$ is nontrivial) or if:

1. $\langle \sigma_1, \ldots, \sigma_{n-2} \rangle$ is a string $C^+$-group,
2. $\langle \sigma_2, \ldots, \sigma_{n-1} \rangle$ is a string $C^+$-group, and
3. $\langle \sigma_1, \ldots, \sigma_{n-2} \rangle \cap \langle \sigma_2, \ldots, \sigma_{n-1} \rangle = \langle \sigma_2, \ldots, \sigma_{n-2} \rangle$.

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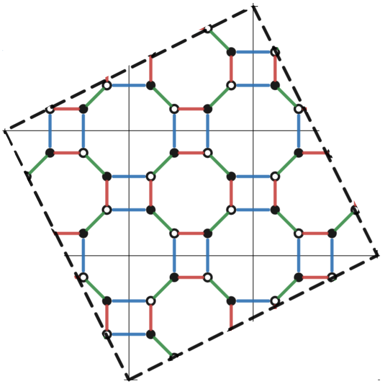
If $\mathcal{P}$ is a chiral *polytope*, then its automorphism group is a string $C^+$-group.

# Groups of chiral polytopes

We can build a polytope from a string $C^+$-group; it might be regular or chiral. It will be chiral if and only if there is no group automorphism that sends $\sigma_1$ to $\sigma_1^{-1}$ and $\sigma_2$ to $\sigma_1^2\sigma_2$ while fixing every other $\sigma_i$.

(These transformations come from looking at the action of the automorphism group on the second flag orbit.)

# Example: $\{4,4\}_{(2,1)}$



The automorphism group is

$$\langle \sigma_1, \sigma_2 \mid \sigma_1^4 = \sigma_2^4 = (\sigma_1\sigma_2)^2 = (\sigma_2^{-1}\sigma_1)^2(\sigma_2\sigma_1^{-1}) = 1 \rangle.$$

From the group, we know that this is chiral because there is no group automorphism that sends $\sigma_1$ to $\sigma_1^{-1}$ and $\sigma_2$ to $\sigma_2^{-1}$.

# Smallest regular covers

Every chiral polytope $\mathcal{P}$ has a unique smallest regular maniplex that covers it. Here is one way to build it:

1. Start with $\Gamma(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$
2. Find the group $\Gamma(\overline{\mathcal{P}}) = \langle \sigma_1', \ldots, \sigma_{n-1}' \rangle$, whose presentation is obtained from $\Gamma(\mathcal{P})$ by changing $\sigma_1$ to $(\sigma_1')^{-1}$, changing $\sigma_2$ to $(\sigma_1')^2 \sigma_2'$, and changing $\sigma_i$ to $\sigma_i'$ for $i \geq 3$.
3. Then take their *parallel product*[3]:

$$\langle \alpha_1, \ldots, \alpha_{n-1} \rangle \leq \Gamma(\mathcal{P}) \times \Gamma(\overline{\mathcal{P}}),$$

   where $\alpha_i = (\sigma_i, \sigma_i')$.

The result is a 'regular' rotation group.

[3] Steve Wilson, *Parallel products in groups and maps*, 1994

# When is the smallest regular cover a polytope?

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**Theorem (Monson, Pellicer, and Williams 2014)**

*If the facets and/or vertex-figures of the chiral polytope $\mathcal{P}$ are regular, then the smallest regular cover of $\mathcal{P}$ is a polytope.*

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Are there examples of chiral polytopes where the smallest regular cover is not a polytope?

# The smallest rank 4 example

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^6 = \sigma_2^9 = \sigma_3^6 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1,$$
$$\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^4, \sigma_3^2\sigma_2 = \sigma_2^4\sigma_3^2\rangle.$$

Here it is in the GAP package RAMP (developed with Mark Mixer and Gordon Williams):

```
gap> P := RotaryManiplex([6,9,6], "s2 s1^2 = s1^2 s2^4, s3^2 s2 = s2^4 s3^2");
RotaryManiplex([ 6, 9, 6 ], "s2 s1^2 = s1^2 s2^4, s3^2 s2 = s2^4 s3^2")
gap> IsPolytopal(P);
true
gap> NumberOfFlags(P);
648
gap> R := SmallestReflexibleCover(P);
reflexible 4-maniplex
gap> NumberOfFlags(R);
1944
gap> IsPolytopal(R);
false
gap> L := SmallChiral4Polytopes([1..647]);;
gap> Size(L);
13
gap> ForAny(L, M -> not(IsPolytopal(SmallestReflexibleCover(M))));
false
```

# 5 families of examples

| $\{p, q, r\}$ | $\sigma_2^{-1}\sigma_1$ | $\sigma_2\sigma_1^{-1}$ | $\sigma_3^{-1}\sigma_2$ | $\sigma_3\sigma_2^{-1}$ | Notes |
|---|---|---|---|---|---|
| $\{2m, m^\alpha, 2m\}$ | $\sigma_1^3\sigma_2^{1+km^{\alpha-1}}$ | $\sigma_1^{-3}\sigma_2^{-1+km^{\alpha-1}}$ | $\sigma_2^{-1+km^{\alpha-1}}\sigma_3^{-3}$ | $\sigma_2^{1+km^{\alpha-1}}\sigma_3^3$ | $m$ odd prime, $\alpha \geq 2$, $1 \leq k \leq m-1$ |
| $\{8, 2^\beta, 8\}$ | $\sigma_1^3\sigma_2^{1+\epsilon_1 2^{\beta-2}}$ | $\sigma_1^{-3}\sigma_2^{-1+\epsilon_1 2^{\beta-2}}$ | $\sigma_2^{-1+\epsilon_2 2^{\beta-2}}\sigma_3^{-3}$ | $\sigma_2^{1+\epsilon_2 2^{\beta-2}}\sigma_3^3$ | $\beta \geq 5, \epsilon_1, \epsilon_2 = \pm 1$ |
| $\{2^{\beta-1}, 2^\beta, 2^{\beta-1}\}$ | $\sigma_1^{-1+2^{\beta-2}}\sigma_2^{-3+\epsilon_1 2^{\beta-2}}$ | $\sigma_1^{1+2^{\beta-2}}\sigma_2^{3+\epsilon_1 2^{\beta-2}}$ | $\sigma_2^{3+\epsilon_2 2^{\beta-2}}\sigma_3^{1+2^{\beta-2}}$ | $\sigma_2^{-3+\epsilon_2 2^{\beta-2}}\sigma_3^{-1+2^{\beta-2}}$ | $\beta \geq 5, \epsilon_1, \epsilon_2 = \pm 1$ |
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[4] Gabe Cunningham and Daniel Pellicer, *Tight chiral polytopes*, 2021

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(But how do I know that these are all examples?...)[4]

[4] Gabe Cunningham and Daniel Pellicer, *Tight chiral polytopes*, 2021

# Chirality groups

Each chiral polytope also has a unique largest regular quotient. The *chirality group*[5] of $\mathcal{P}$, denoted $X(\mathcal{P})$, is the kernel of that projection.

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Example: The chirality group of $\{4,4\}_{(2,1)}$ is cyclic of order 5, generated by $\sigma_2^{-1}\sigma_1$.

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# Rank 4

**Theorem (C., 2023)**

*Suppose that $\mathcal{P}$ is a chiral 4-polytope with chiral facets $\mathcal{K}$ and chiral vertex-figures $\mathcal{L}$. Then the smallest regular cover of $\mathcal{P}$ is a polytope if and only if $X(\mathcal{K})$ and $X(\mathcal{L})$ have trivial intersection when considered as subgroups of $\Gamma(\mathcal{P})$.*

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Note that $X(\mathcal{K}) \leq \langle \sigma_1, \sigma_2 \rangle$ and $X(\mathcal{L}) \leq \langle \sigma_2, \sigma_3 \rangle$, and so if $X(\mathcal{K}) \cap X(\mathcal{L})$ is nontrivial, then it is contained in $\langle \sigma_2 \rangle$. So:

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*If $\mathcal{K}$ is a chiral polyhedron such that $X(\mathcal{K})$ has trivial intersection with $\langle \sigma_2 \rangle$, then every chiral 4-polytope with facets isomorphic to $\mathcal{K}$ has a polytopal smallest regular cover.*

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Example: All chiral 4-polytopes with facets isomorphic to $\{4, 4\}_{(2,1)}$ have a polytopal smallest regular cover.

# General rank

**Theorem (C., 2023)**

*Suppose that $\mathcal{P}$ is a chiral n-polytope with chiral facets $\mathcal{K}$ and chiral vertex-figures $\mathcal{L}$. Let $\mathcal{M}$ be the vertex-figures of $\mathcal{K}$ (= the facets of $\mathcal{L}$). Then the smallest regular cover of $\mathcal{P}$ is a polytope if and only if $X(\mathcal{K}) \cap X(\mathcal{L}) \leq X(\mathcal{M})$, when considered as subgroups of $\Gamma(\mathcal{P})$.*

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**Corollary**

*If $\mathcal{K}$ is a chiral $(n-1)$-polytope such that $X(\mathcal{K})$ has trivial intersection with $\langle \sigma_2, \ldots, \sigma_{n-1} \rangle$, then every chiral n-polytope with facets isomorphic to $\mathcal{K}$ has a polytopal smallest regular cover.*

Thank you!

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*If $\mathcal{P}$ is a chiral polytope with totally chiral facets and vertex-figures, then its smallest regular cover is a polytope if and only if the vertex-figures of the facets of $\mathcal{P}$ are also totally chiral.*